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INFLUENCE OF EDGE CONDITIONS ON THE STABILITY OF AXIALLY COMPRESSED CYLINDRICAL SHELLS

by B. O. Almroth

Prepared under Contract No. NAS 1-3778 *by*
LOCKHEED MISSILES AND SPACE COMPANY
Sunnyvale, Calif.
for

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FOREWORD

The work described in this report was carried out under Contract NAS 1-3778 with the National Aeronautics and Space Administration.

This is a revision of the original report, which was issued in August 1964.

SUMMARY

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Recent investigations by Stein and by Fischer on the influence of edge conditions on the critical load of cylindrical shells are here extended to cover six additional combinations of boundary conditions. The results show that drastic reductions of the critical load for cylinders with lateral support of the edges are obtained only if the edges are free in the tangential direction. For other boundary conditions, this reduction is never more than about 20 percent. Consequently, the results of this investigation alone cannot explain the well-known discrepancy between theory and test data. However, the importance of the choice of boundary conditions for practical analysis is clearly demonstrated.

Author

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NOTATION

A_1, A_2	integration constants; see Eqs. (19) and (20)
D	$Et^3/[12(1 - \nu^2)]$
E	Young's modulus
F	stress function; see Eq. (22)
F_m	value of F at $x = x_m$
L	shell length
M	total number of discrete points
N	compressive axial load per unit width
$\bar{N}$	$N/(2\gamma)$
N_x, N_y, N_{xy}	axial, circumferential, and shear forces per unit width
W	lateral displacement; see Eq. (22)
W_m	value of W at $x = x_m$
Z	$L^2(1 - \nu 2)^{1/2}/(rt)$
a_1	$k(1 + \bar{N})^{1/2}$
a_2	$k(1 - \bar{N})^{1/2}$
f	stress function
f_0, f_1	parts of stress function, prebuckling and incremental
h	distance between neighboring discrete points
k	$(2\gamma)^{-1/2}$
n	number of waves in circumferential direction
p	internal pressure

$\bar{p}$	$(pr)/(Et)$
r	shell radius
t	shell thickness
u, v, w	axial, circumferential, and lateral displacements
w_0, w_1	parts of lateral displacement, prebuckling and incremental
x, y	axial and circumferential coordinates
x_m	value of axial coordinate at discrete point
y_m	see Eq. (28)
α_1	$(a_1L)/(2r)$
α_2	$(a_2L)/(2r)$
$\epsilon_x, \epsilon_y, \gamma_{xy}$	axial, circumferential, and shear strain
γ	$(t/r)/[12(1 - \nu^2)]^{1/2}$
ν	Poisson's ratio
σ_{CR}	critical axial stress
σ_{CL}	classical value of σ_{CR}
∇^4	$\partial^4/\partial x^4 + 2\partial^4/\partial x^2 \partial y^2 + \partial^4/\partial y^4$

When subscripts x and y follow a comma, they indicate partial differentiation of the principal variable with respect to x or y. Primes indicate total derivatives with respect to x.

Section 1
INTRODUCTION

Serious disagreement between the results of classical theory and experiments on the buckling of axially compressed cylindrical shells has long been known to exist. Despite considerable effort, this problem is still not completely understood. New light was thrown on the matter in the recent investigation by Stein reported in Refs. 1 and 2. In that analysis, the deformations and stresses induced by edge support were for the first time rigorously taken into account. It was shown that previous inconsistent assumptions in the classical theory with regard to boundary conditions in the prebuckling state can significantly affect the theoretical buckling load.

The analysis of Ref. 1 is limited to one specific set of simple-support boundary conditions. Fischer, in Ref. 3, later considered the same problem but used different in-plane boundary conditions. The results of the two analyses differ significantly, and it appeared desirable that a more complete investigation be undertaken to establish the effect of edge conditions on the classical buckling load for axially compressed cylindrical shells. Such an analysis is presented here and includes cylinders with clamped as well as simply supported edges. Eight different sets of boundary conditions are considered, and the approach to the problem is largely the same as in Ref. 1. The prebuckling displacements are obtained through rigorous solution of the applicable differential equations. As incremental displacements are assumed to be infinitesimal, the buckling equations are linear but have variable coefficients. Solutions to these are obtained by use of a finite difference technique.

It was pointed out by Koiter (Ref. 4) that the classical solution of the buckling problem for axially compressed cylinders is based on the boundary conditions used by Fischer, and that Stein's condition of zero tangential edge force may lead to a reduced critical load even if a membrane solution is used for prebuckling displacements. This was shown to be the case by Ohira (Ref. 5) and by Hoff and Rehfield (Ref. 6). Therefore, the present analysis includes, for comparison, solutions for the case in which effects of edge restrain on prebuckling configurations have been omitted. This case will be referred to as the membrane prebuckling solution.

Section 2
BASIC EQUATIONS

As in Ref. 1, Donnell-type equations are used in the analysis. In their nonlinear form the three equilibrium equations are

$$N_{x,x} + N_{xy,y} = 0$$
$$N_{y,y} + N_{xy,x} = 0 \qquad (1)$$
$$D\nabla^4 w + N_y/r - (N_x w_{,xx} + N_y w_{,yy} + 2N_{xy} w_{,xy}) - p = 0$$

The corresponding relations between stresses, strains, and displacements are

$$N_x = [Et/(1 - \nu^2)] (\epsilon_x + \nu\epsilon_y)$$
$$N_y = [Et/(1 - \nu^2)] (\epsilon_y + \nu\epsilon_x) \qquad (2)$$
$$N_{xy} = [Et/2(1 + \nu)] \gamma_{xy}$$

$$\epsilon_x = u_{,x} + 1/2\, w_{,x}^2$$
$$\epsilon_y = v_{,y} + w/r + 1/2\, w_{,y}^2 \qquad (3)$$
$$\gamma_{xy} = u_{,y} + v_{,x} + w_{,x} w_{,y}$$

The first two of the equilibrium equations are identically satisfied if a stress function, f , is introduced such that

$$N_x = f_{,yy} \; ; \; N_y = f_{,xx} \; ; \; N_{xy} = - f_{,xy} \tag{4}$$

By use of Eqs. (4), the third equilibrium equation can now be written

$$D\nabla^4 w + f_{,xx}/r - (f_{,yy} w_{,xx} + f_{,xx} w_{,yy} - 2f_{,xy} w_{,xy}) - p = 0 \tag{5}$$

Through elimination of u and v in Eqs. (2) and (3), a compatibility equation can be derived:

$$(1/Et) \nabla^4 f - w_{,xx}/r - w_{,xy}^2 + w_{,xx} w_{,yy} = 0 \tag{6}$$

Equations (5) and (6) are two simultaneous differential equations in the two unknown variables f and w . Together with an adequate set of boundary conditions, they govern the behavior of the shell. A total of eight boundary conditions – four at each edge of the cylinder – is required. As the displacements are expected to be symmetrical about the midpoint of the shell, it is also possible to use a set of four symmetry conditions together with four conditions at the edge of the shell.

The axial coordinate is assumed to be zero at the midpoint and consequently

$$\begin{aligned} \text{at } x = 0 : \quad w_{,x} &= 0 \\ w_{,xxx} &= 0 \\ f_{,x} &= 0 \\ f_{,xxx} &= 0 \end{aligned} \tag{7}$$

The shell is assumed to be supported in the radial direction at the edge. Therefore,

$$\text{at } x = L/2: \; w = 0 \tag{8}$$

Three additional conditions are needed and will be obtained by selection of one of each of the following three pairs:

$$\begin{aligned} \text{at } x = L/2: \; w_{,x} &= 0 \quad \text{or} \quad w_{,xx} = 0 \\ u &= 0 \quad \text{or} \quad N_x = 0 \\ v &= 0 \quad \text{or} \quad N_{xy} = 0 \end{aligned} \tag{9}$$

Evidently, eight different combinations are possible and these, in combination with Eqs. (7) and (8), form eight complete sets of boundary conditions. All of these sets will be considered in the analysis.

For shorter writing, the different combinations will be referred to as follows:

Case S1	when	$w_{,xx} = 0$,	$u = 0$,	$v = 0$
S2		$w_{,xx} = 0$,	$N_x = 0$,	$v = 0$
S3		$w_{,xx} = 0$,	$u = 0$,	$N_{xy} = 0$
S4		$w_{,xx} = 0$,	$N_x = 0$,	$N_{xy} = 0$
C1		$w_{,x} = 0$,	$u = 0$,	$v = 0$
C2		$w_{,x} = 0$,	$N_x = 0$,	$v = 0$
C3		$w_{,x} = 0$,	$u = 0$,	$N_{xy} = 0$
C4		$w_{,x} = 0$,	$N_x = 0$,	$N_{xy} = 0$

Section 3
PREBUCKLING DISPLACEMENTS

Displacements and stresses will be composed such that

$$
\begin{aligned}
w &= w_0 + w_1 \\
f &= f_0 + f_1
\end{aligned}
\tag{10}
$$

where subscript 0 identifies the conditions at impending buckling and hence subscript 1 identifies incremental quantities.

At impending buckling the deformations are axially symmetrical and therefore the equation of equilibrium can be written

$$
Dw_{0,xxxx} + f_{0,xx}/r - f_{0,yy}w_{0,xx} - p = 0 \tag{11}
$$

If the applied compressive axial load is denoted by N ,

$$
\begin{aligned}
f_{0,yy} &= - N \\
f_{0,xx} &= Et\,w_0/r - \nu N
\end{aligned}
\tag{12}
$$

Substitution of Eqs. (12) into Eqs. (11) yields

$$
Dw_{0,xxxx} + Nw_{0,xx} + Etw_0/r^2 - (\nu N/r + p) = 0 \tag{13}
$$

The solution to this equation can be written in the form

$$w_0 = r(2\nu\gamma\overline{N} + \overline{p})(1 + A_1 \sin a_1 x \sinh a_2 x + A_2 \cos a_1 x \cosh a_2 x) \quad (14)$$

where

$$a_1 = k(1 + \overline{N})^{1/2}$$

$$a_2 = k(1 - \overline{N})^{1/2}$$

$$k = (2\gamma)^{-1/2}$$

$$\gamma = (t/r)/\left[12(1 - \nu^2)\right]^{1/2}$$

$$\overline{N} = N/(2\gamma)$$

$$\overline{p} = (pr)/(Et) \quad (15)$$

The integration constants are obtained through substitution of Eq. (4) into the boundary conditions. For simply supported shells, these are

$$\text{at} \quad x = L/2: \; w_0 = 0 \quad , \quad w_{0,xx} = 0 \quad (16)$$

and for clamped shells

$$\text{at} \quad x = L/2: \; w_0 = 0 \quad , \quad w_{0,x} = 0 \quad (17)$$

The following notations are introduced.

$$\alpha_1 = (a_1 L)/(2r)$$

$$\alpha_2 = (a_2 L)/(2r) \quad (18)$$

For simply supported shells the solution will be given by

$$A_1 = - \left[(1 - \overline{N}^2)^{1/2} \sin \alpha_1 \sinh \alpha_2 + \overline{N} \cos \alpha_1 \cosh \alpha_2 \right] /$$

$$\left[(1 - \overline{N}^2)^{1/2} \left(\cosh^2 \alpha_2 - \sin^2 \alpha_1 \right) \right]$$

$$A_2 = - \left[(1 - \overline{N}^2)^{1/2} \cos \alpha_1 \cosh \alpha_2 - \overline{N} \sin \alpha_1 \sinh \alpha_2 \right] /$$

$$\left[(1 - \overline{N}^2)^{1/2} \left(\cosh^2 \alpha_2 - \sin^2 \alpha_1 \right) \right] \qquad (19)$$

For clamped shells,

$$A_1 = - 2 \left[(1 - \overline{N})^{1/2} \cos \alpha_1 \sinh \alpha_2 - (1 + \overline{N})^{1/2} \sin \alpha_1 \cosh \alpha_2 \right] /$$

$$\left[(1 + \overline{N})^{1/2} \sinh 2\alpha_2 + (1 - \overline{N})^{1/2} \sin 2\alpha_1 \right]$$

$$A_2 = - 2 \left[(1 - \overline{N})^{1/2} \sin \alpha_1 \cosh \alpha_2 + (1 + \overline{N})^{1/2} \cos \alpha_1 \sinh \alpha_2 \right] /$$

$$\left[(1 + \overline{N})^{1/2} \sinh 2\alpha_2 + (1 - \overline{N})^{1/2} \sin 2\alpha_1 \right] \qquad (20)$$

Section 4
BUCKLING EQUATIONS

The equations needed for the solution of the buckling problem are obtained through substitution of Eqs. (10) into Eqs. (5) and (6). From the equations so derived, the prebuckling equations can be subtracted and higher order terms in the infinitesimal incremental displacements omitted. The equilibrium and compatibility equations may then be written in the form

$$
\begin{aligned}
D\nabla^4 w_1 + f_{1,xx}/r + N w_{1,xx} - w_{0,xx} f_{1,yy} + (\nu N - Etw_0/r) w_{1,yy} &= 0 \\
(1/Et)\nabla^4 f_1 - w_{1,xx}/r + w_{0,xx} w_{1,yy} &= 0
\end{aligned}
\tag{21}
$$

The equations may be separated with respect to the space variables by use of the substitution

$$
\begin{aligned}
f_1 &= F(x) \sin(ny/r) \\
w_1 &= W(x) \sin(ny/r)
\end{aligned}
\tag{22}
$$

The conditions of continuity in the circumferential direction will be satisfied if n, the number of circumferential waves, is an integer. In substitution of Eqs. (22) into Eqs. (21), primes are used to denote differentiation with respect to x.

$$
\begin{aligned}
D[W'''' - 2(n/r)^2 W'' + (n/r)^4 W] + F''/r + NW'' + (n/r)^2 w_0'' F \\
- (n/r)^2 (\nu N - Etw_0/r) W = 0
\end{aligned}
\tag{23}
$$

$$
(1/Et)[F'''' - 2(n/r)^2 F'' + (n/r)^4 F] - W''/r - (n/r)^2 w_0'' W = 0
$$

These are linear differential equations with variable coefficients. Besides the trivial solution $W = F = 0$, solutions which satisfy the boundary conditions exist for particular values of N. The lowest of these values represents the critical load of the cylinder.

The boundary conditions are given in Eqs. (7), (8), and (9). The conditions pertaining to w, f, N_x, or N_{xy} are converted in terms of W and F in an obvious way. The conditions for u and v may be expressed in terms of W and F by use of Eqs. (2), (3), (4), and (22).

It is found that the condition $v = 0$ can be replaced by

$$F'' + \nu(n/r)^2 F = 0 \tag{24}$$

and the condition $u = 0$ by

$$F''' - (2 + \nu)(n/r)^2 F' - EtW' = 0 \tag{25}$$

The classical formulation of the problem is obtained if the membrane solution

$$w_0 = (\nu Nr)/(Et) + (pr^2)/(Et) \tag{26}$$

is substituted in Eqs. (23).

The simplified equations so obtained for the case of hydrostatic pressure were solved by Sobel in Ref. 7 for the eight sets of boundary conditions considered here. Corresponding solutions presented here for the case of axial compression were obtained by use of Sobel's computer program. These solutions are compared with solutions of Eqs. (23).

Section 5
METHOD OF SOLUTION

Solutions to the equations governing the buckling problem are obtained through use of the finite difference method. The equations are approximately satisfied at a number of discrete points along the shell length in the interval $0 \le x \le L/2$. In order that the eight boundary conditions may be defined, four additional points, two at each end, are located outside the interval. The points are chosen to be equally spaced, and the distance between adjacent points is denoted by h . The total number of points, hence, is

$$M = L/(2h) + 5$$

The axial coordinates of the points are given by

$$x_m = h(m - 3) \quad ; \quad m = 1, 2, 3, \ldots, M$$

The values of the stress function and the lateral displacement at the point with the axial coordinate x_m are referred to as F_m and W_m . At each of the points, F_m and W_m are unknown and hence the total number of unknowns is

$$2M = (L/h + 10)$$

At each of the interior points $(m = 3, 4 \ldots, M - 2)$ the compatibility and equilibrium equations are formulated in terms of finite differences, yielding $2[L/(2h) + 1] = L/h + 2$ equations. The boundary conditions, written in terms of finite differences, provide eight equations for a total of $(L/h + 10)$.

The following difference approximations are used:

$$y_i' = [y_{i+1} - y_{i-1}]/(2h)$$

$$y_i'' = [y_{i+1} - 2y_i + y_{i-1}]/h^2$$

$$y_i''' = [1/2\, y_{i+2} - y_{i+1} + y_{i-1} - 1/2\, y_{i-2}]/h^3$$

$$y_i'''' = [y_{i+2} - 4y_{i+1} + 6y_i - 4y_{i-1} + y_{i-2}]/h^4 \tag{27}$$

In order that nonzero elements in the coefficient matrix will be concentrated around the main diagonal, the first four equations are the boundary conditions at midlength, and the last four the boundary conditions at the edge. Furthermore, of the remaining equations, the odd numbered are compatibility equations and the even numbered are equilibrium equations. The unknowns are denoted y_i, such that

$$y_{2m} = W_m$$

$$y_{2m-1} = F_m$$

where

$$1 \leq m \leq L/(2h) + 5 \tag{28}$$

It follows from Eqs. (27) and (28) that

$$F_m' = [y_{2m+1} - y_{2m-3}]/(2h)$$

$$F_m'' = [y_{2m+1} - 2y_{2m-1} + y_{2m-3}]/h^2$$

$$F_m''' = [1/2\, y_{2m+3} - y_{2m+1} + y_{2m-3} - 1/2\, y_{2m-5}]/h^3$$

$$F_m'''' = [y_{2m+3} - 4y_{2m+1} + 6y_{2m-1} - 4y_{2m-3} + y_{2m-5}]/h^4$$

$$W'_m = [y_{2m+2} - y_{2m-2}]/(2h)$$

$$W''_m = [y_{2m+2} - 2y_{2m} + y_{2m-2}]/h^2$$

$$W'''_m = [1/2\, y_{2m+4} - y_{2m+2} + y_{2m-2} - 1/2\, y_{2m-4}]/h^3$$

$$W''''_m = [y_{2m+4} - 4y_{2m+2} + 6y_{2m} - 4y_{2m-2} + y_{2m-4}]/h^4 \qquad (29)$$

By use of Eqs. (29), the boundary conditions at midlength may be written

$$\text{Equation 1} \;;\quad F'_3 = (y_7 - y_3)/2h = 0$$

$$2 \;;\quad F'''_3 = (1/2\, y_9 - y_7 + y_3 - 1/2\, y_1)/h^3 = 0$$

$$3 \;;\quad W'_3 = (y_8 - y_4)/2h = 0$$

$$4 \;;\quad W'''_3 = (1/2\, y_{10} - y_8 + y_4 - 1/2\, y_2)/h^3 = 0 \qquad (30)$$

The compatibility conditions provide the equations numbered 5 , 7 , 9 , . . . , 2M-5 .

For $m = 3, 4, 5, \ldots, M-2$,

$$(1/Et)\Big[(y_{2m+3} - 4y_{2m+1} + 6y_{2m-1} - 4y_{2m-3} + y_{2m-5})/h^4$$
$$- 2(n/r)^2 (y_{2m+1} - 2y_{2m-1} + y_{2m-3})/h^2 + (n/r)^4 y_{2m-1}\Big]$$
$$- (1/r)(y_{2m+2} - 2y_{2m} + y_{2m-2})/h^2 - (n/r)^2 (w''_0)_m\, y_{2m} = 0$$

(31)

Similarly the equations numbered 4 , 6 , 8 , . . . , 2M-4 are obtained from equilibrium conditions.

For $m = 3, 4, 5, \ldots, M-2$,

$$D\Big[(y_{2m+4} - 4y_{2m+2} + 6y_{2m} - 4y_{2m-2} + y_{2m-4})/h^4 - 2(n/r)^2(y_{2m+2} - 2y_{2m} + y_{2m+2})/h^2 + (n/r)^4 y_{2m}\Big] + (1/r)(y_{2m+1} - 2y_{2m-1} + y_{2m-3})/h^2 + N(y_{2m+2} - 2y_{2m} + y_{2m-2})/h^2 + (n/r)^2 (w_0'')_m y_{2m-1} - (n/r)^2 \Big[\nu N - Et(w_0)_m\Big] y_{2m} = 0 \tag{32}$$

The last four boundary conditions in terms of finite differences are

Equation 2M-3: $\quad W = 0 \ ; \ y_{2M-4} = 0$

Equation 2M-2: $\quad W_{,x} = 0 \ ; \ (y_{2M-2} - y_{2M-6})/2h = 0$

or $\quad W_{,xx} = 0 \ ; \ (y_{2M-2} - 2y_{2M-4} + y_{2M-6})/h^2 = 0$

Equation 2M-1: $\quad u = 0 \ ; \ (1/2\, y_{2M-1} - y_{2M-3} + y_{2M-7} - 1/2\, y_{2M-9})/h^3$

$\qquad - (2 + \nu)(n/r)^2 (y_{2M-3} - y_{2M-7})/(2h)$

$\qquad - Et(y_{2M-2} - y_{2M-6})/(2h) = 0$

or $\quad N_x = 0 \ ; \ y_{2M-5} = 0$

Equation 2M: $\quad v = 0 \ ; \ (y_{2M-3} - 2y_{2M-5} + y_{2M-7})/(h^2) + \nu(n/r)^2 y_{2M-5} = 0$

or $\quad N_{xy} = 0 \ ; \ (y_{2M-3} - y_{2M-7})/2h = 0 \qquad (33)$

Once the boundary conditions have been chosen, Eqs. (30) through (33) define a linear homogeneous equation system of the order 2M. This equation system can have nontrivial solutions only if the determinant of the coefficient matrix equals zero. The lowest value of N for which the determinant is zero represents the critical load of the cylinder. This value is found through computation of the determinant for a series of values of the load such that the solution can be found graphically.

For the computations, an IBM-7094 digital computer was used. In programming, advantage was taken of the fact that elements sufficiently far off the diagonal were zero. Furthermore, the computer program provided a possibility to alternately reduce the matrix and compute new coefficients. In this way the available storage space did not limit the number of points that could be used. Therefore, the limit was determined by round-off errors, and a double precision procedure was used.

All numerical results were obtained for a value of Poisson's ratio equal to 0.3.

Section 6
RESULTS

The accuracy of numerical results, of course, depends on the use of a sufficient number of points in the finite difference scheme. On the other hand, the computer time increases rapidly with an increasing number of points. The convergence of the method, therefore, was explored through calculation of critical loads for fixed shell parameters and a successively increasing number of points. Some of the results so obtained are shown in Table 1. By use of these exploratory calculations it was possible to establish, as a function of the parameter Z, the number of points needed for 1/2 percent, or better, accuracy in the final results.

Computed critical values of the axial load for the case of zero lateral pressure are shown in Table 2. For comparison, corresponding results were also obtained by use of the membrane prebuckling solution. It was found that with boundary conditions corresponding to cases S3 and S4 ($W_{,xx} = 0$, $N_{xy} = 0$) these results differ very little from the results in Table 2. In the other six cases the critical load with membrane prebuckling solution, within the parameter range considered, is equal to or insignificantly higher than the classical buckling load for simply supported cylinders ($\overline{N} = 1.0$) .

It appears from Table 2 that, with the exception of very short shells ($Z = 50$) , the critical load is practically independent of the parameters r/t and L/r in all cases. In contrast to expectations, lower values of the critical load are in some cases found for the very short shells. Therefore, the influence of the shell length on the critical load was studied in more detail. The critical load versus the parameter L/r , for a cylinder with $r/t = 100$ and with boundary conditions corresponding to case C2, is shown in Fig. 1. Here the number of circumferential waves was held constant ($n = 8$) . For long shells the critical load is independent of shell length, and for very short shells the critical load is, as expected, monotonically increasing with decreasing shell length. In the intermediate range an oscillatory behavior is displayed. It may

Table 1

CONVERGENCE OF CRITICAL LOAD

(a) $r/t = 10^2$, $L/r = 0.7$, $\bar{p} = 0$

$L/(2h)$	$\overline{N}_{CR}$	
	Case S3	Case S4
50	0.5150	0.5100
100	.5136	.5085
150	.5133	.5083
200	.5132	.5082
250	.5132	.5082
300	.5132	.5082

(b) $r/t = 10^2$, $L/r = 3.2$, $\bar{p} = 0$

$L/(2h)$	$\overline{N}_{CR}$	
	Case S3	Case S4
100	0.5153	0.5155
200	.5118	.5114
300	.5109	.5108
400	.5106	.5105
500	.5104	.5103
600	.5103	.5103
700	.5102	.5102
800	.5102	.5102

Table 1 (Cont.)

(c) $r/t = 10^4$, $L/r = 0.32$, $\bar{p} = 0$

L/(2h)	$\bar{N}_{CR}$	
	Case S3	Case S4
120	0.5054	0.5038
240	.5025	.5006
360	.5021	.5004
480	.5019	.5003
600	.5019	.5003
720	.5019	.5003

(d) $r/t = 10^2$, $L/r = 3.2$, $\bar{p} = 0.232$

Case S3	
L/(2h)	$\bar{N}_{CR}$
120	0.679
240	.676
360	.6750
480	.6745
600	.6745
720	.6745

(e) $r/t = 10^2$, $L/r = 0.7$, $\bar{p} = 0$

L/(2h)	$\bar{N}_{CR}$	
	Case C1	Case C3
60	0.91015	0.90997
120	.91019	.91002
180	.91020	.91003
240	.91021	.91003
300	.91021	.91003

Table 2

CRITICAL VALUES OF $\overline{N}$ (for $\bar{p} = 0$)

r/t	L/r	$\overline{N}_{CR}$ and (Number of Waves) in Case							
		S1	S2	S3	S4	C1	C2	C3	C4
10^4	0.07	0.872(85)	0.805(79)	0.507(2)	0.499(2)	0.908(83)	0.862(82)	0.908(83)	0.862(82)
↓	0.16	.864(86)	.846(84)	.504(2)	.501(2)	.926(88)	.914(86)	.926(88)	.913(86)
	0.24	.868(86)	.843(79)	.503(2)	.501(2)	.927(87)	.908(78)	.927(87)	.906(78)
10^4	0.32	.867(84)	.844(82)	.503(2)	.501(2)	.926(86)	.910(82)	.926(86)	.908(82)
10^3	0.222	.874(27)	.807(25)	.507(2)	.500(2)	.907(27)	.863(25)	.907(27)	.861(26)
↓	0.506	.863(27)	.846(27)	.505(2)	.501(2)	.927(27)	.914(27)	.927(27)	.913(27)
	0.760	.867(27)	.842(25)	.503(2)	.502(2)	.930(28)	.908(25)	.930(28)	.906(25)
10^3	1.014	.867(27)	.844(26)	.503(2)	.502(2)	.926(27)	.910(26)	.926(27)	.908(26)
10^2	0.7	.876(9)	.806(8)	.513(2)	.508(2)	.910(9)	.863(8)	.910(9)	.862(8)
↓	1.6	.858(8)	.849(8)	.512(2)	.511(2)	.929(9)	.917(9)	.929(9)	.916(9)
	2.4	.868(9)	.843(8)	.511(2)	.511(2)	.930(9)	.908(8)	.930(9)	.903(8)
10^2	3.2	.868(9)	.844(8)	.510(2)	.510(2)	.928(9)	.911(8)	.928(9)	.909(8)

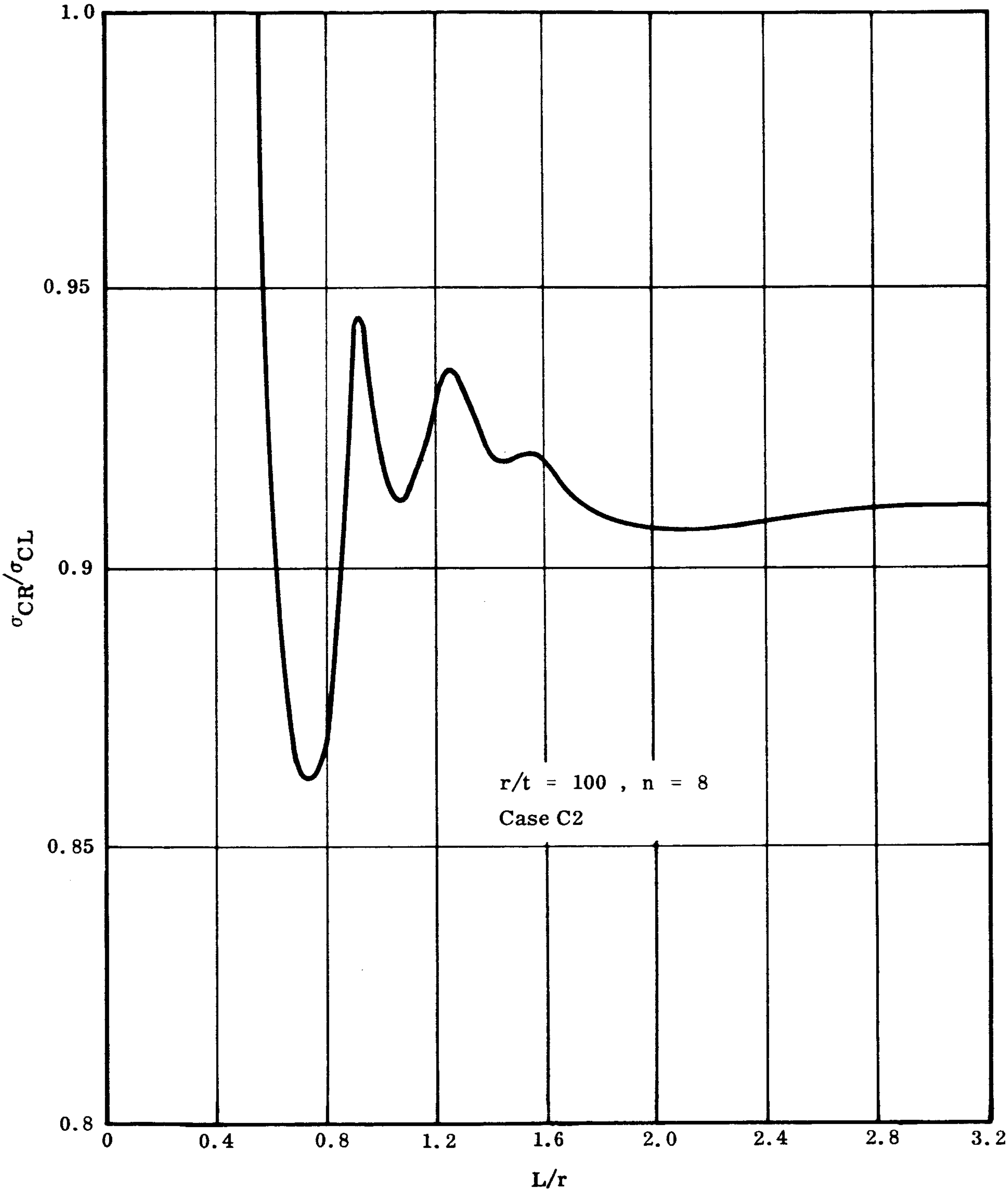


Fig. 1 Influence of Shell Length for Clamped Cylinders

be seen that the deep minimum, which is reached just before the critical load increases monotonously, is very close to $L/r = 0.7$.

A similar variation of the critical load with L/r was found for the cases S3 and S4. In Fig. 2a the critical load is shown versus L/r for case S3. It is seen that in this case the general behavior does not change when the influence of lateral restraint in the prebuckling analysis is neglected. For relatively long cylinders the curve with membrane prebuckling solution is slightly below the rigorous solution. However, for very short cylinders this difference increases, as may be expected. In Fig. 2b the length dependence is shown for a cylinder with boundary conditions corresponding to case S4. In both these cases the curves are valid for a fixed number of waves $(n = 2)$.

Critical values of the external pressure with zero axial load were also computed for the eight different sets of boundary conditions. In this case the results agree with those reported in Ref. 7, except for very short cylinders. The slight discrepancy for short shells was observed for simply supported shells also in Ref. 8.

Critical loads under combined loading are shown in Figs. 3, 4, 5, and 6. In cases S1 and S2 (Fig. 3) the value of r/t has practically no influence on critical combinations of the axial stress and external pressure parameters, within the range of geometrical parameters under consideration.

The analysis with the membrane prebuckling solution indicates that the difference in critical loads for cases S3 and S4 is negligible for all values of the pressure parameter. However, in the presence of internal pressure the rigorous solution gives different results. This is shown by the interaction curves in Figs. 4 and 5. Here the number of waves corresponding to minimum critical load generally is two, but, for higher values of the pressure, in case S3, this minimum occasionally occurs at a larger number of waves.

In case S3 the curves show that for most combinations of geometrical parameters there exists a range of the pressure parameter within which three solutions are obtained. Of course, when the axial load on the shell is increased under constant internal pressure,only the lowest of these solutions is meaningful.

For clamped cylinders, it was found again that the parameter r/t has no influence on critical combinations of the axial stress and pressure parameters. It was found also that tangential restraint at the edge does not affect the critical load. Interaction curves for clamped cylinders are shown in Fig. 6.

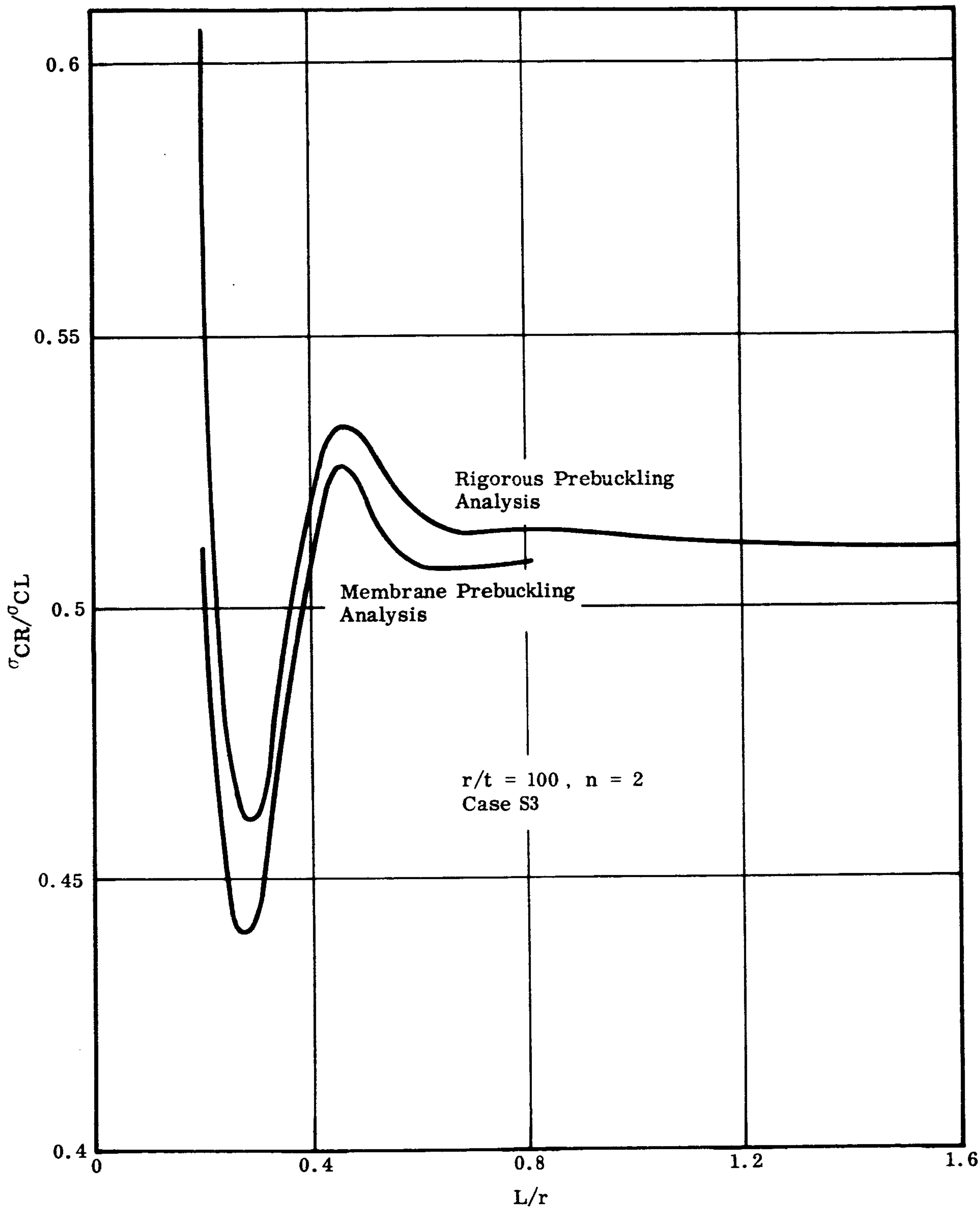


Fig. 2 a Influence of Shell Length for Simply Supported Cylinders

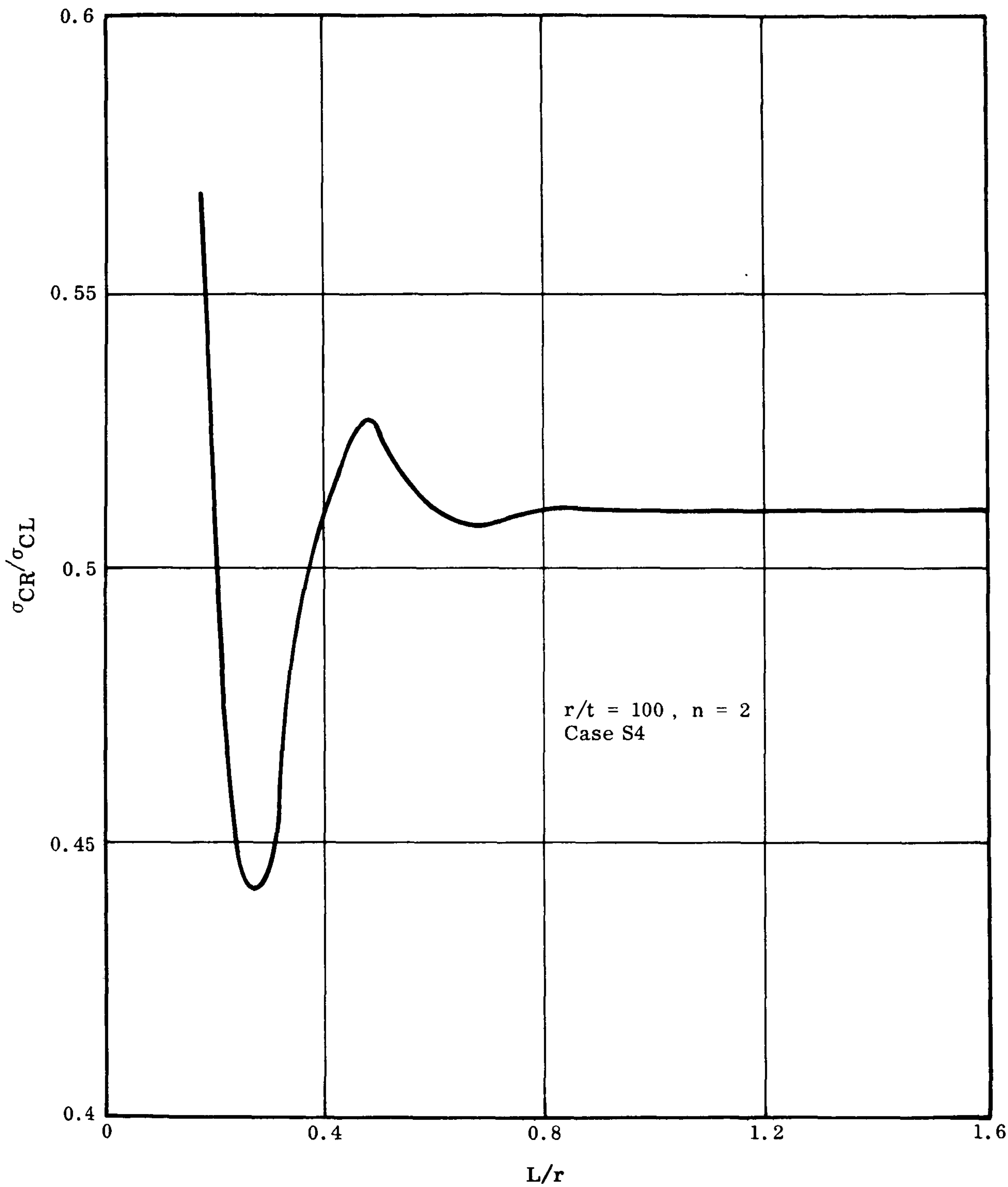


Fig. 2b Influence of Shell Length for Simply Supported Cylinders

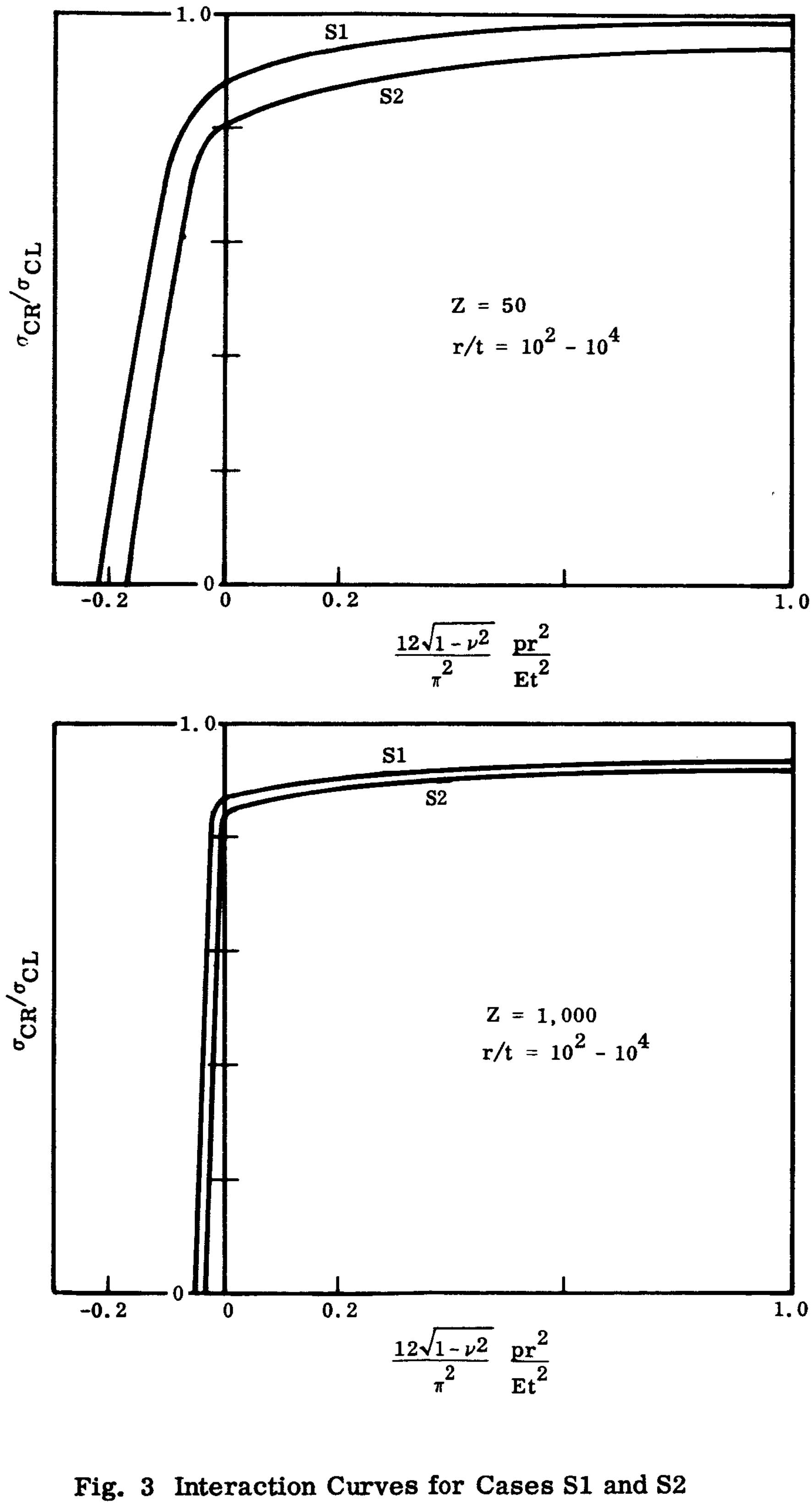


Fig. 3 Interaction Curves for Cases S1 and S2

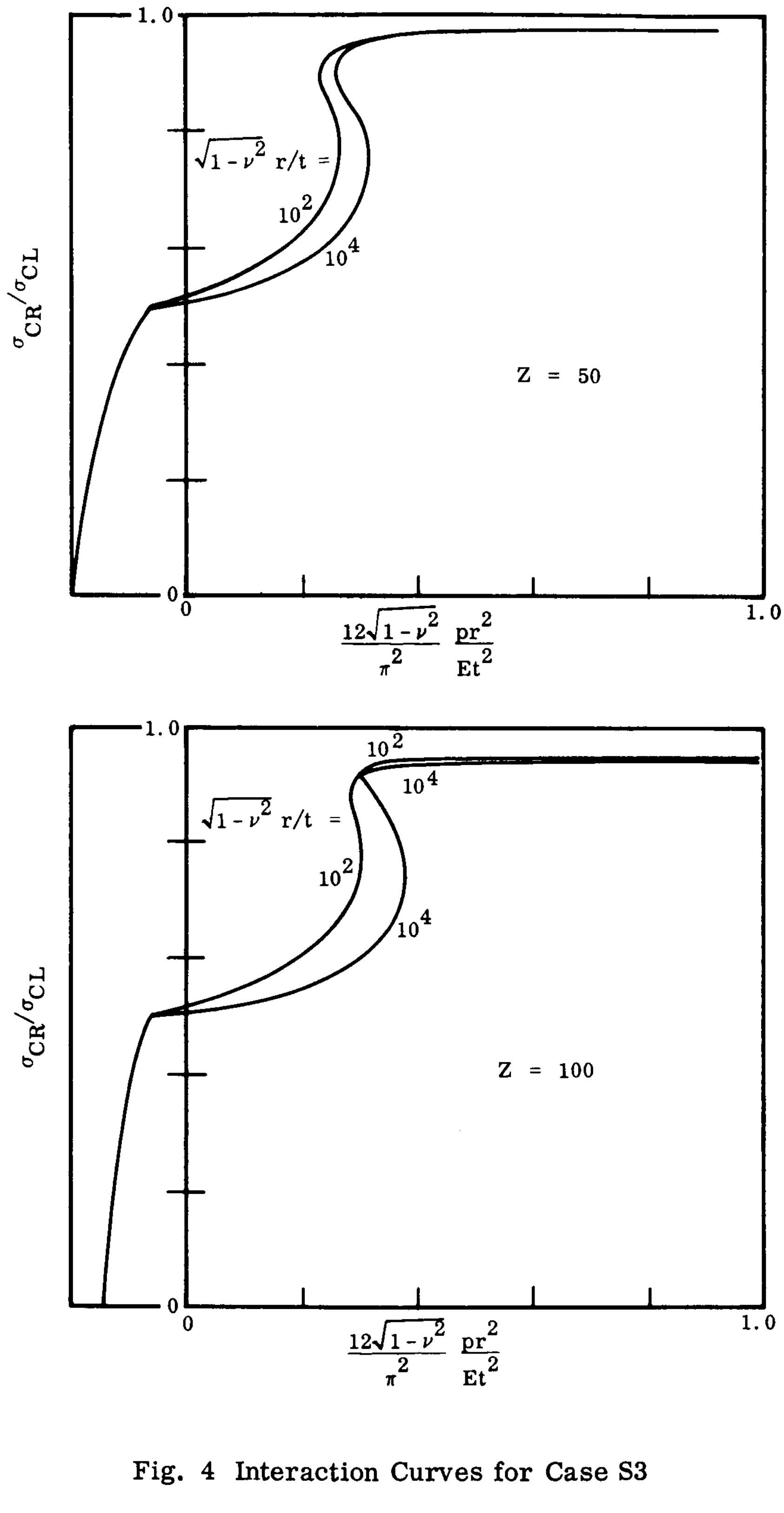
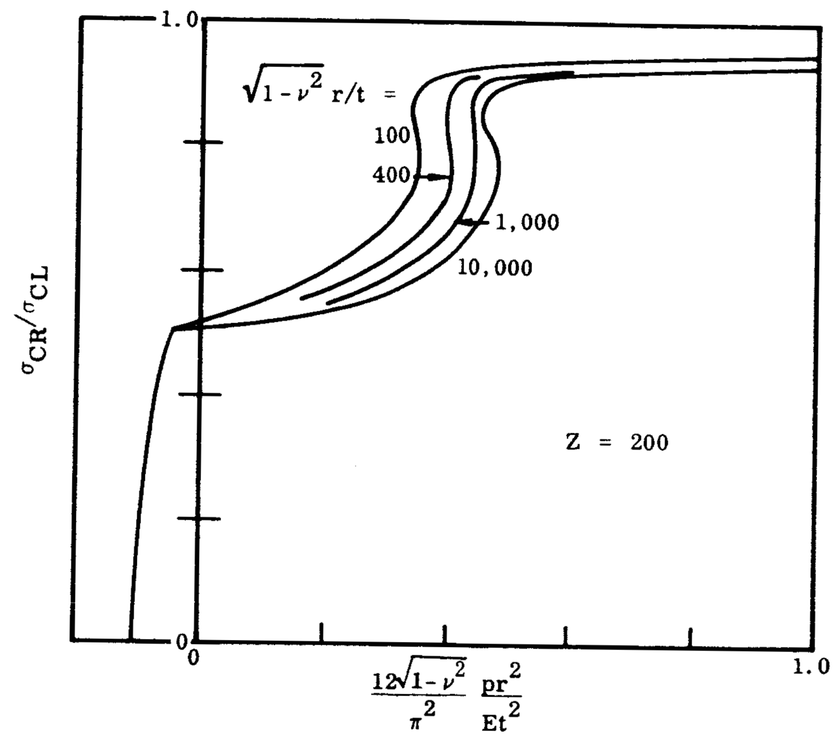


Fig. 4 Interaction Curves for Case S3



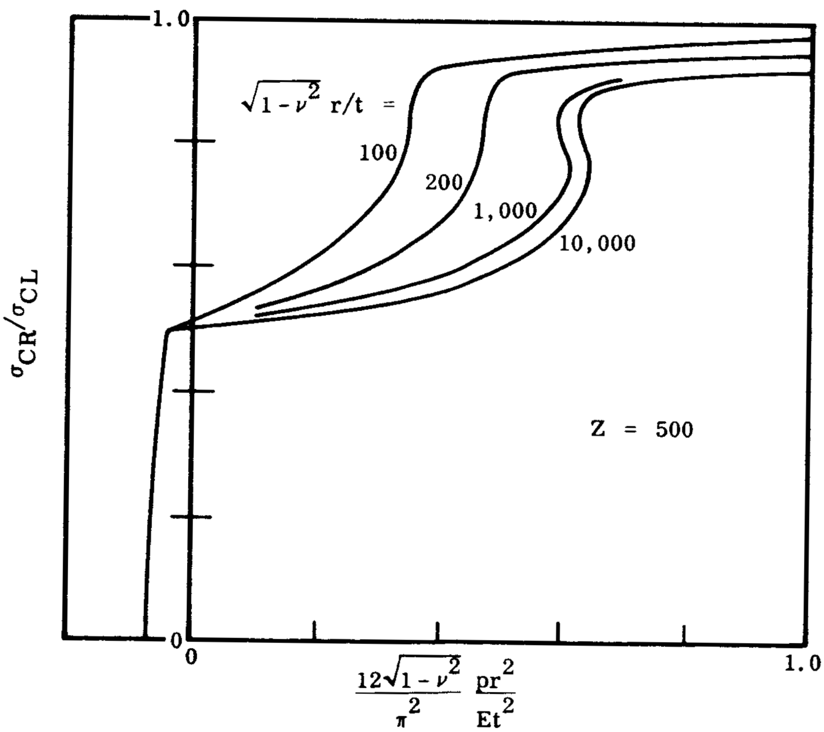


Fig. 4 Interaction Curves for Case S3 (cont.)

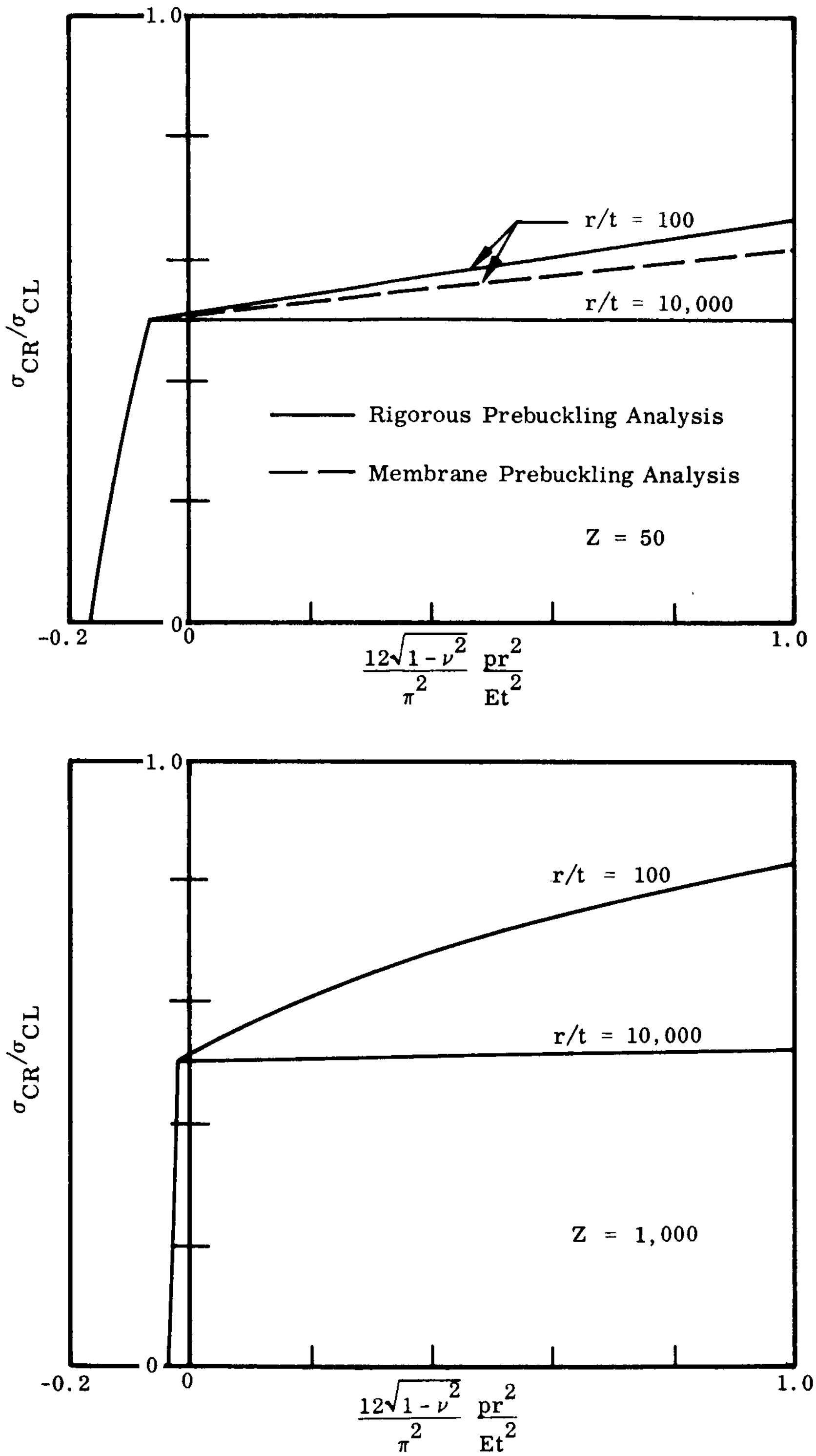


Fig. 5 Interaction Curves for Case S4

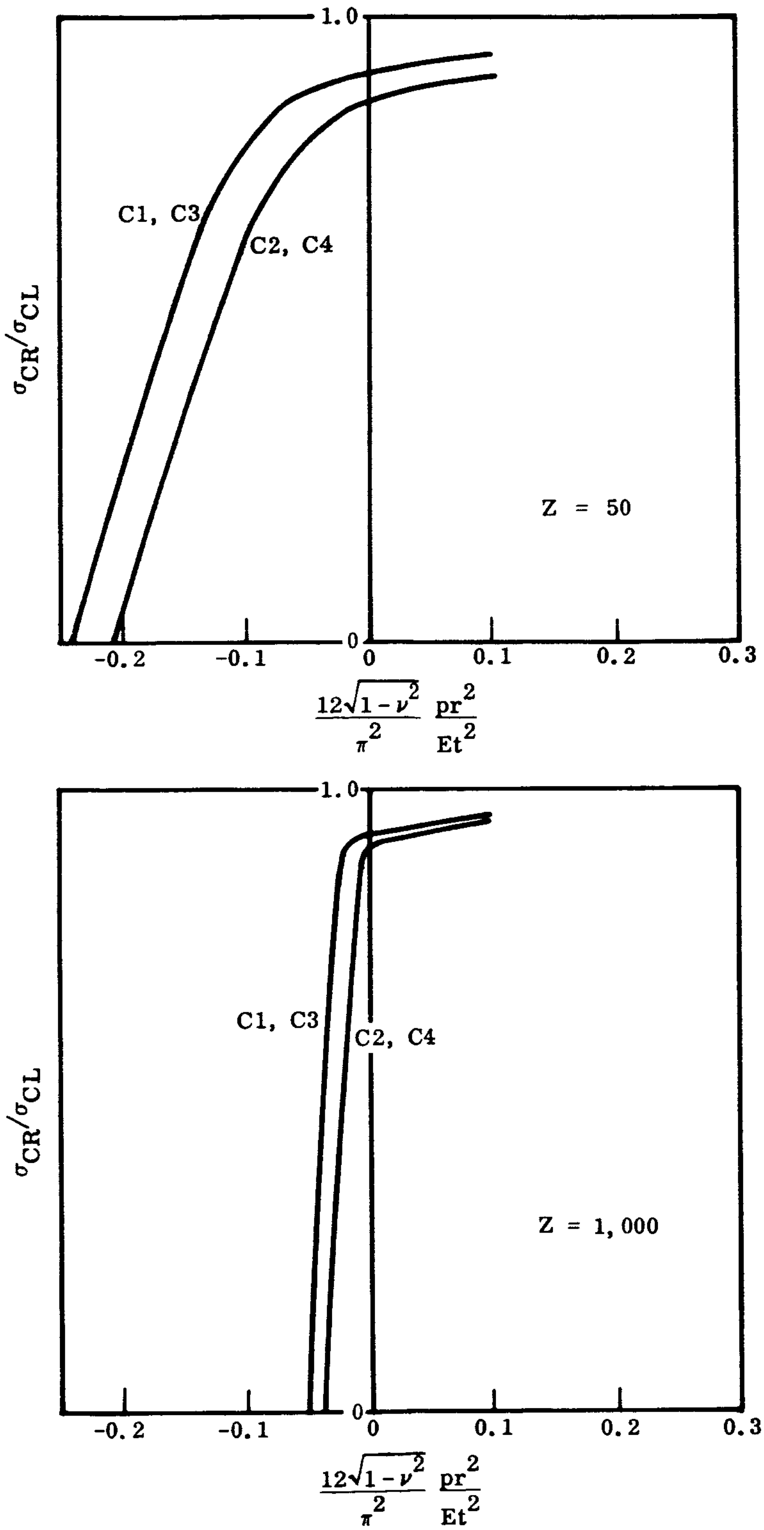


Fig. 6 Interaction Curves for Clamped Cylinders, $p \leq 0$, $10^2 \leq r/t \leq 10^4$

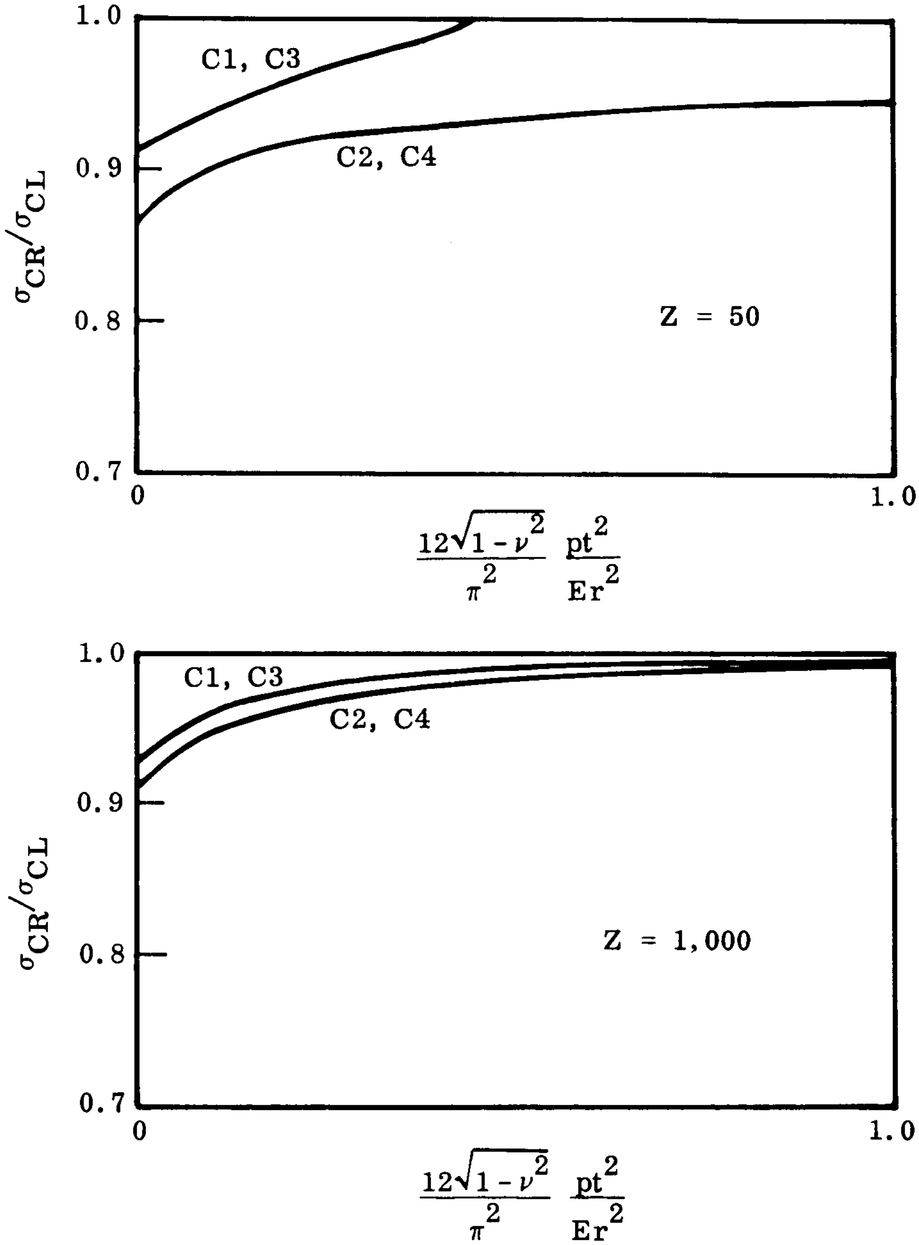


Fig. 6 Interaction Curves for Clamped Cylinders, $p \leq 0,\ 10^2 \leq r/t \leq 10^4$ (cont.)

Section 7
CONCLUSIONS

The results obtained here are in agreement with the results of the example considered by Fischer, and also, over most of the parameter ranges, in reasonably close agreement with the results presented by Stein. However, for very thin and long cylinders the difference between present results and those presented by Stein is appreciable. Therefore, a computer program based on Eq. (15) of Ref. 2 was derived. It was found that if a sufficiently fine mesh size was used, the numerical results were in good agreement with the present results throughout the parameter range.

The results for simply supported cylinders with the edge free to move in the tangential direction show a drastic reduction of the critical load in comparison with the classical value. For relatively long, nonpressurized cylinders, this reduction is practically independent of whether an accurate or a membrane prebuckling solution is used. For pressurized cylinders and for very short cylinders, the rigorous analysis gives higher values of the critical load.

For simply supported cylinders in the case of zero tangential edge displacement, and for clamped cylinders, use of the rigorous analysis leads to a reduction of the critical load. This reduction is rather moderate in comparison with the discrepancy between theory and tests. It appears that these boundary conditions generally are applicable in experiments, and therefore the edge effects studied here cannot alone explain the discrepancy between theory and tests. A final solution of this problem will probably have to include effects of initial geometric inaccuracies.

The results for cylinders with edges free in the tangential direction, first obtained by Stein, are somewhat related to the results by Nachbar and Hoff in Ref. 9. In Ref. 9 the cylinder edge was considered to be free in the lateral direction, and also in this case the critical load was drastically reduced. Although neither of these conditions is realized in experimental analysis, they are still interesting in that it is feasible that in practical applications elastic restraint at the edges may be rather weak.

Section 8
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